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Chaotic behavior of the piccolo

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Abstract

A direct numerical solution of the Navier-Stokes equations in three dimensions has been used to compute the sound pressure produced by a piccolo as a function of time *p(t).* For moderate blowing speeds *u*, a pure tone is produced, but as *u* is increased *p(t)* exhibits an increasingly complex behavior. The behavior of *p(t)* is consistent with a positive Lyapunov exponent at high values of *u*. Detailed results for the power spectrum reveal a simple pure tone dominated by a single frequency at low *u*, as expected. As *u* is increased additional frequencies appear in the spectrum along with broadband noise in certain spectral regions. The results suggest that the piccolo is, under certain blowing conditions, a chaotic system.

Keywords: piccolo, instrument modelling, wind instruments, chaos

Chaotic behavior of the piccolo

1 Introduction

Much progess has been made in recent years in the modelling of musical instruments at a "first principles" level. By this we mean modelling based on the fundamental laws of mechanics. We now have models of this kind for many string instruments such as the piano and guitar (see, e.g., [1-4]). Wind instruments present a bigger challenge than string instruments, since a detailed description of a wind instrument requires application of the Navier-Stokes equations. These are a set of nonlinear partial differential equations that describe the behaviour of the air flowing through an instrument. For realistic instrument geometries solution of the Navier-Stokes equations requires a numerical treatment with a high performance (parallel) computer, and in the past several years available computers have become powerful enough to yield useful results for wind instruments, although early work of this kind was first given more than a decade ago (e.g., [5,6]). In this paper we describe such a simulation study of the piccolo. We present results for the acoustic pressure as a function of time *p(t)* outside the instrument for different values of the blowing speed. The results suggest that at high blowing speeds the behaviour of *p(t)* can be chaotic.

2 The model

The Navier-Stokes equations in three dimensions are given in Eq. 1. Here *u*, *v*, and *w* are the components of the air velocity along the *x*, *y*, and *z*, directions and ρ is the density, *c* is the speed of sound and ν is the viscosity. Note that we must assume a compressible fluid in order to calculate the sound pressure. For an ideal gas such as air, the sound pressure *p* (which is the variation of the pressure from its equilibrium value) is proportional to variations of the density from its background value.

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}-\nu\nabla^2 u=0\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial y}-\nu\nabla^2 v=0\\
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{c^2}{\rho}\frac{\partial \rho}{\partial z}-\nu\nabla^2 w=0
\end{aligned}
\tag{1}
$$

Our model is shown in Figure 1. This is a simplified model of a piccolo, in which the body of the instrument is a cylindrical tube of total length 270 mm and diameter 12.5 mm, with a rectangular

embouchure opening of length 10 mm (along the axis) and width 7 mm. The distance from the center of the embouchure to the closed end was 11 mm. There are no tone holes and the embouchure is a straight hole through the tube with no lip plate.

The lips of the player were modelled as a solid channel closely adjacent to the embouchure as shown in Figure 1. To "blow" the flute, we imposed a constant air flow at the end of the lip channel farthest from the embouchure and parallel to the channel. The channel curves as it approaches the embouchure so as to direct its air jet at an angle with respect to the direction tangent to the top of the tube. For all of the results presented in this paper that angle was 20º. This blowing angle was found to give a "well behaved" result for *p(t)* for a wide range of blowing speeds. Some other blowing angles were also investigated and those results will be presented elsewhere. These dimensions and the overall design of our model are similar those of a real piccolo. We expect the behaviour we calculate to also be found in the transverse flute.

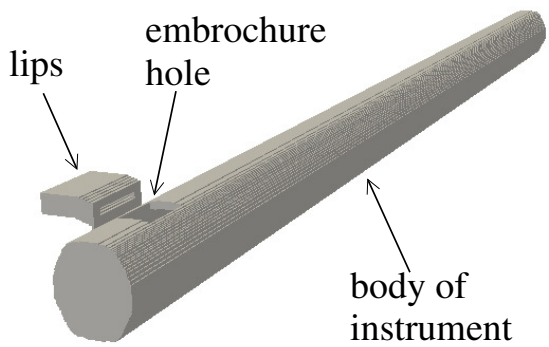

Figure 1. Model piccolo showing the location and orientation of the channel that acts as the lips. The end of the tube at the lower left is closed while the end at the upper right is open. The dimensions are given in the text.

The Navier-Stokes equations were solved in a non-uniform Cartesian grid using an explicit finite difference time domain algorithm described elsewhere [7]. Inside and near the piccolo the grid spacing was 0.2 mm perpendicular to the axis and 0.5 mm along the axis, and the time step was 0.2 μs. The piccolo was contained in a closed rectangular box with dimensions 60x60x360 mm^3 and with a total of $2x10^7$ grid points.

3 Results

3.1 Time dependence of the acoustic pressure

Figure 2 shows typical results for the acoustic pressure as a function of time *p(t)*, as calculated for a point outside the piccolo. In Fig. 2a the blowing pressure was *u* = 14 m/s, and the behaviour of *p(t)* was regular and periodic. As a musical tone, it was nearly a pure tone with only a small amount of power at the second harmonic (the spectrum will be shown below). Figure 2a actually shows the behaviour with *u* = 14 m/s with two slightly different blowing conditions. The solid curve shows results with the blowing speed ramped up linearly with time starting from zero and reaching the final value of *u* = 14 m/s at *t* = 5.00 ms (a ramp-up time that seems typical for wind instruments.) The dotted curve in Fig. 2a shows results with a ramp-up time of 5.05 ms. It is seen that this 1% change in the ramp-up time produced very little change in the resulting *p(t)*.

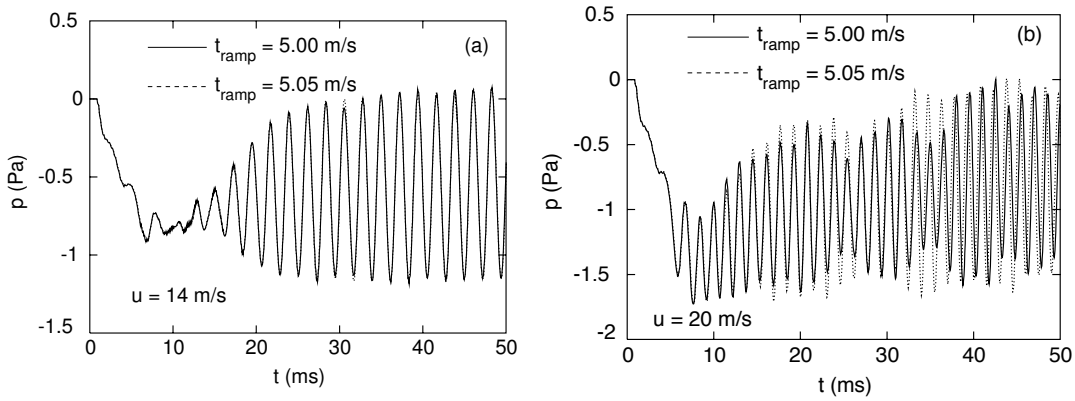

Figure 2: Results for the sound pressure as a function of time, *p(t)*, at a location outside the tube and away from the tube axis in Fig. 1. Left: For a blowing speed *u* = 14 m/s. Right: For a blowing speed *u* = 20 m/s. The solid and dotted curve show results for two different ramp-up times as explained in the text.

Figure 2b shows results for a larger blowing speed *u* = 20 m/s. For this blowing speed *p(t)* has a more complex spectrum, and deviates from a pure tone in ways that will be described in detail below. In addition, changing the ramp-up time from 5.00 ms to 5.05 ms produces a very significant change in *p(t)*. This sensitivity to a small change in the blowing conditions seen in Fig. 2b suggests that when blown at *u* = 20 m/s led us to consider if this model piccolo is a chaotic dynamical system [8,9].

A central property of a chaotic dynamical system is an extreme sensitivity to changes in the initial conditions. For our model piccolo, the movement of the state of the piccolo through phase space can be described by *p(t)*. If we then consider two phase space trajectories $p_1(t)$ and $p_2(t)$ that originate from two slightly different initial conditions, then for a chaotic system the two trajectories diverge exponentially with time, with an exponent λ called the Lyapunov exponent

[10]. The Lyapunov exponent is positive for a system in a chaotic state, and zero or negative for a non-chaotic system. An analysis of our model piccolo suggests that λ undergoes a transition from a negative to a positive value for blowing speeds around 15 m/s [11].

The piccolo would not be the first musical instrument to exhibit chaotic behaviour. Previous work on the clarinet [12-14] and the cymbal [15] has found that these two instruments can also be chaotic. In the case of the clarinet the nonlinearity of the reed is responsible for the chaotic behaviour, while for the cymbal the nonlinear equation of motion of the cymbal is the direct cause. For the piccolo, the chaotic behaviour seems to be due to the inherent properties (i.e., nonlinearities) of the Navier-Stokes equations. This is not surprising in view of the work of Lorenz [5] and many other analyses of fluid systems described by the Navier-Stokes equations.

3.2 Spectral properties

Our results suggest that there is a transition to a chaotic state as the blowing speed is increased. It is believed that there are a relatively small number of different ways that this transition can occur. Well known routes to chaos include period doubling, intermittancy, and a route involing Hopf bifurcations [10]. These different routes to chaos can be distinguished through the spectrum of the trajectory through phase space; in our system this trajectory is specified by *p(t)*.

Figure 3 shows results for the power spectrum of *p(t)* at two blowing speeds, u = 14 and 20 m/s, the same blowing speeds considered in Fig. 2. Examination of these spectra and the spectra for other blowing speeds reveals the following. At low blowing speeds *p(t)* is approximately a pure tone, dominated by a component with a fundamental frequency f_1 and with a much smaller component at f_2, where $f_2 = 2\ f_1$. The component at f_2 is thus simply the second harmonic of f_1, which is what would be expected for a typical piccolo tone. This behavior is seen in Fig. 3a with u = 14 m/s.

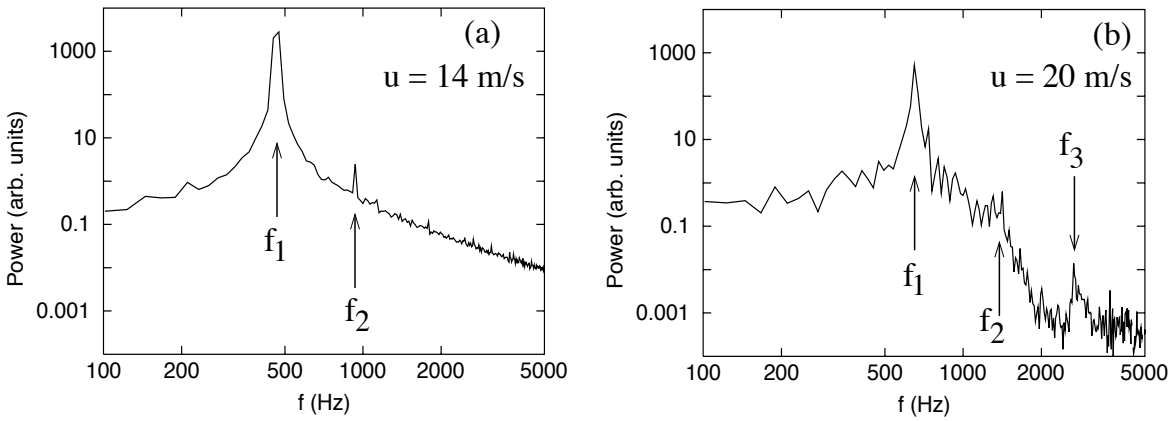

Figure 3: Results for the spectrum of *p(t)* at the two blowing speeds considered in Fig. 2.The approximate locations of the fundamental frequency f_1, the second harmonic f_2, and a third frequency component f_3 that is not harmonically related to f_1 are shown. While the vertical scale is arbitrary, it is the same in parts (a) and (b).

As the blowing speed is increased, the component at f_1 broadens and the component at f_2 also broadens and reduces in amplitude, becoming a shoulder on a smooth background. This is illustrated in Fig. 3b with u = 20 m/s. In addition, a third component appears at a higher frequency f_3.

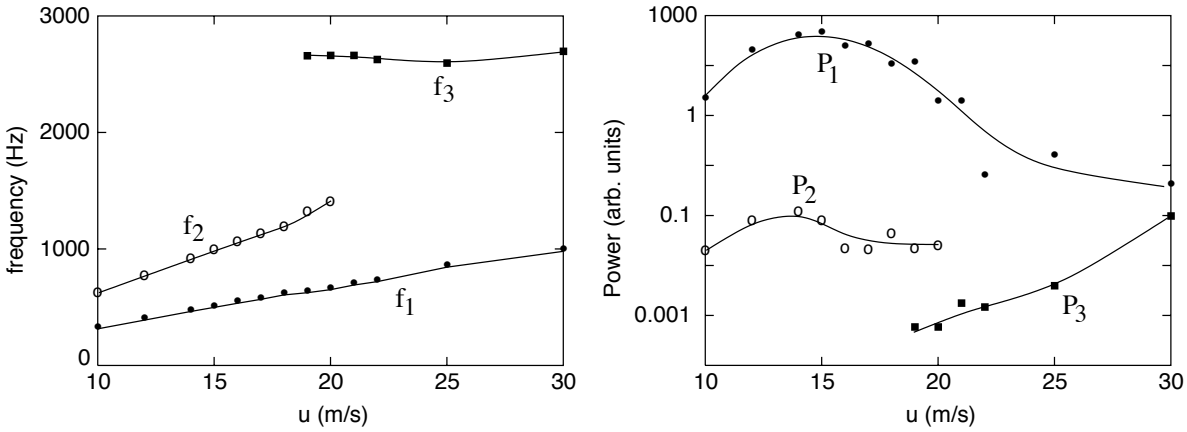

Figure 4: Left: Frequencies f_1, f_2, and f_3 of the components identified in the power spectra in Fig. 3 as a function of blowing speed u. Right: Peak power of the components at frequencies f_1, f_2, and f_3. as a function of blowing speed.

Figure 4 shows how the frequencies of these three components and their peak powers vary as a function of blowing speed. The second harmonic at f_2 disappears into the background around u = 20 m/s while the component at f_3 becomes visible only at a slightly lower blowing speed. Note that we are showing the peak power in Fig. 4; the total integrated power under these peaks would exhibit a slightly different behaviour, since the peak widths change considerably with u. However, allowing for variations of the width would not change the qualitative behaviour seen in Fig. 4.

4 Discussion

The current understanding of chaotic dynamical systems is that there are a relatively small number of routes to chaos [10]. One of those routes involves a Hopf bifurcation in the system's trajectory in phase space, to a state in which an oscillation with a particular frequency is found. As the system is driven harder, i.e., by varying a parameter such as the blowing speed in our case, the system may then undergo a series of Hopf bifurcations with each corresponding to the appearance of a new mode with its own frequency. Theoretical arguments [9,10] suggest that after two or three bifurcations the system enters a chaotic state with two or more independent frequencies. This picture is broadly consistent with the behavior seen in the spectra in Fig. 3 and the behavior of the modes found in Fig. 4.

5 Conclusions

The results for the spectra suggest that our model piccolo is a chaotic dynamical system in which the transition to the chaotic state is governed by a Hopf bifurcation. Our results are broadly similar to that found in other systems described by the Navier-Stokes equations [16]. We should note that an analysis of the sensitivity of *p(t)* to changes in the initial conditions and the associated Lyapunov exponent [11] suggests the transition to chaos occurs at a blowing speed of about u = 15 m/s, which is near where the fundamental component exhibits a peak power in Fig. 4.

There is one aspect of our model piccolo that is a bit unexpected. The variation of f_1 with blowing speed in Fig. 4a is much stronger that expected or found for a real piccolo [17]. For a real instrument one finds that f_1 increases only a little with blowing speed, usually much less than a semitone, before jumping an octave. Our model piccolo shows a much stronger variation of f_1. The reason for this is not entirely clear, but some preliminary measurements on piccolos with different shaped embouchures reveal that embouchures with shaper edges, like those in our model (Fig. 1) can exhibit a quite sizeable variation of f_1 before jumping to a higher octave. This will require further study, as will the possibility of chaotic behaviour of *p(t)* produced by a real instrument.

Acknowledgments

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