Symmetry breaking and topology of acoustic waves in phononic structures

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Abstract

We present examples of phononic structures that break four types of symmetry, namely time-reversal symmetry, parity symmetry, chiral symmetry and particle-hole symmetry. The implications of symmetry breaking on the topology of the acoustic wave function in the space of its Eigen values are discussed. Particular attention is focused on the torsional topology of acoustic waves in periodic media in wave vector space. Two types of approach to achieve symmetry breaking are considered: (a) intrinsic topological phononic structures where symmetry breaking occurs from the internal structural characteristics, and (b) extrinsic topological phononic structures where external stimuli such as spatio-temporal modulations of the physical properties of the medium are used to break symmetry.

Keywords: phononic structure, symmetry breaking, topology
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1 Introduction

Broken symmetry (be it time-reversal, parity, chiral, or particle-hole symmetry) underpins the concepts of non-reciprocal wave propagation. Dissipation is a textbook example of broken time-reversal symmetry [1], where waves are damped along the positive time line, but amplified with a change in the sign of the time variable—thus broken symmetry. Single broken symmetries, such as those due to dissipation, while being critical to non-reciprocity, are alone insufficient to enable non-reciprocal wave propagation [2]. In contrast, combining broken symmetries, such as is provided by nonlinearity, dispersion, dissipation, or chiral structure, has been shown to lead to such non-reciprocal propagation [3,4]. Indeed, a major current open challenge is to realize non-reciprocal, solid-state acoustic wave propagation with a high rectification ratio (large contrast between two propagation directions), while maintaining the shape and frequency content of the original signal [2].

The factors leading to non-reciprocity can be elucidated through the concepts of topological order, and the closely related concept of topological protection. Symmetry breaking is linked to constraints on the topological form of acoustic wave functions. For instance, in the context of topology, for a damped oscillator, the amplitude of the wave function has properties isomorphic to the parallel transport of a vector along a strip-like manifold that represents frequency space, with a twist at the oscillator resonant frequency (and a \( \pi \)-phase shift accompanying an amplitude sign change as it crosses resonance). Dissipation aside, dispersion is one of the most central elements to topological order, and thus non-reciprocity. A simple, linear one-dimensional (1D) harmonic monoatomic chain is a dispersive system—but one that obeys time-reversal symmetry, happens to be reciprocal, and supports waves with relatively simple topologies. In contrast, a 1D anharmonic monoatomic chain may create resonant nonlinear phonon modes [3] that are dispersive, but whose generation efficiency depends on wave number and amplitude. In this case, the interplay between the nonlinearity and dispersion of the system leads to non-conventional topologies [5,6] that can create non-reciprocal wave propagation [3].

Two-classes of phononic structures exist possessing non-conventional topology, namely intrinsic and extrinsic systems [7-25]. In intrinsic systems, time-reversal symmetry is broken through internal mechanisms such as chirality without adding external energy. For extrinsic topological systems, energy is added to break symmetry. Thus far, the topologically protected propagation of acoustic waves in intrinsic systems has been limited to the edges of two- (2D) and three-dimensional (3D) materials. These topologically protected edge states have been shown to lead to unidirectional, non-reciprocal, propagation along the materials' edges or surfaces. Here, we illustrate examples of topologically constrained symmetry breaking phononic structures which possess either or both intrinsic and extrinsic symmetry breaking mechanisms.
2 Intrinsic phononic structures

Figure1: Phononic structure composed of mass and springs which modes are represented in the long wavelength limit by a Klein-Gordon equation.

The elastic displacement $u$ of the phononic structure illustrated in figure 1 obeys in the long wavelength limit a relativistic Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - \beta^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u = 0$$

with $\alpha^2 = K_I/m$ and $\beta^2 = K_0/m$. $K_0$ and $K_I$ are the stiffness of the longitudinal and side springs. $m$ is the mass of the spheres. The spacing between masses is taken as unity.

This equation can be written as the set of Dirac-like equations for "particle" $\Psi$ and "antiparticle/hole," $\Psi$:

$$\begin{bmatrix} \sigma_x \frac{\partial}{\partial t} + i\beta \sigma_y \frac{\partial}{\partial x} - iat \end{bmatrix} \Psi = 0; \quad \begin{bmatrix} \sigma_x \frac{\partial}{\partial t} + i\beta \sigma_y \frac{\partial}{\partial x} + iat \end{bmatrix} \Psi = 0$$

where $\sigma_x$ and $\sigma_y$ are the 2x2 Pauli matrices: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $I$ is the 2x2 identity matrix. The quasi-standing wave modes of this phononic structure do not satisfy individually time reversal symmetry ($t \rightarrow -t$), parity symmetry ($x \rightarrow -x$), nor chiral symmetry (directions of propagation are not equivalent) as shown below by the spinorial properties of the modes.

Writing solutions in the form:

$$\Psi_k = \Psi(k, \omega_k) = c_0 \xi_k(k, \omega_k)e^{(\pm)i\omega_k t}e^{(\pm)ikx}$$

and $\Psi_k = \bar{\Psi}(k, \omega_k) = c_0 \bar{\xi}_k(k, \omega_k)e^{(\pm)i\omega_k t}e^{(\pm)ikx}$ where $\xi_k$ and $\bar{\xi}_k$ are two by one spinors and inserting the various forms for these solutions in equations (1a,b) lead to the same Eigen values $\omega = \pm \sqrt{\alpha^2 + \beta^2 k^2}$.

The spinor component of the solutions take the form $\begin{pmatrix} s_1 \sqrt{\omega} \\ s_2 \sqrt{\omega} \end{pmatrix}$ with $s_1$ and $s_2$ having the values +1 or -1 and $\sqrt{\omega} = \sqrt{\omega \pm \beta k}$. The solutions of equation (1a,b) $\Psi$ and $\Psi$ have the non-conventional torsional symmetry properties in the space: $k, \omega$:

$$T_{\omega \rightarrow -\omega} \Psi(k, \omega) \rightarrow \Psi(k, \omega) \quad ;$$

$$T_{\omega \rightarrow -\omega} \bar{\Psi}(k, \omega) \rightarrow i\sigma_x \bar{\Psi}(k, \omega) \quad ;$$

$$T_{\omega \rightarrow -\omega} \bar{\Psi}(k, -\omega) \rightarrow i\sigma_x \bar{\Psi}(k, \omega) \quad ;$$

Where we have defined $T_{\omega \rightarrow -\omega}$ and $T_{\omega \rightarrow -\omega}$ as transformations that change the sign of the frequency and wave number, respectively. As one crosses the origin $k=0$, the multiplicative
factor $i^n$ indicates that the wave function accumulated a phase of $\frac{\pi}{2}$. Also, the Pauli operator $\sigma_x$ enables the transition from the space of solutions $\Psi$ to the space of $\overline{\Psi}$. We also note the orthogonality condition $\overline{\Psi} \sigma_x \Psi = 0$. The spinorial wave functions is therefore supported by a square cross section manifold with quarter-turn torsional topology (Figure 2).

Figure 2: Schematic representation of the manifold supporting $\Psi$ and $\overline{\Psi}$. The manifold exhibits a local quarter-turn twist around $k=0$. The square cross section of the manifold reflects the orthogonality of $\Psi$ and $\overline{\Psi}$. The colored arrows are isomorphic to the wave functions and are parallel transported on the manifold along the direction of wave number. Their change in orientation is indicative of the phase change.

3 Extrinsic phononic structures

We now consider the phononic structure of Figure 1 with application of a spatio-temporal modulation of the parameter $\alpha$. Equations (1a,b) become in the case of a modulation with a general phase $\varphi$:

\[
\begin{align*}
&\left[ \sigma_x \frac{\partial}{\partial t} + i \beta \sigma_y \frac{\partial}{\partial x} - i \alpha_0 l - i \alpha_1 \sin(Kx + \Omega t + \varphi) \right] \Psi = 0 \quad (2a) \\
&\left[ \sigma_x \frac{\partial}{\partial t} + i \beta \sigma_y \frac{\partial}{\partial x} + i \alpha_0 l + i \alpha_1 \sin(Kx + \Omega t + \varphi) \right] \overline{\Psi} = 0 \quad (2b)
\end{align*}
\]

Here, $K = \frac{2\pi}{L}$ where $L$ is the period of the modulation. $\Omega$ is the frequency modulation and its sign determines the direction of propagation of the modulation. Applying the joint time-symmetry ($t$ becomes $-t$) and parity symmetry ($x$ becomes $-x$) to equation (2a) does not result in equation (2b) for all phases $\varphi$ but a few special special values. The modulated Equations (2a,b) have lost the symmetry properties of the unmodulated Dirac equations (Eq. 1a,b).

In recent work [26], we have shown that by applying a sinusoidal spatiotemporal modulation and using multiple time scale perturbation theory leads to a correction of the unperturbed equation (1a) (with zeroth-order corrected solution $\psi^{(0)}$) that takes the form:

\[
\left[ \sigma_x \left( \frac{\partial}{\partial t} - i \Phi_k \cdot \right) + i \beta \sigma_y (ik\cdot - iA_{k\cdot}) - i(\alpha_0 + m_{k\cdot}) l \right] \psi^{(0)}(k\cdot, t) = 0 \quad \text{with} \quad \Phi_k \cdot \text{ and } A_k \cdot \text{ depending on the wave number, } k\cdot \text{ and the characteristics of the modulation (period and velocity). The quantity } \Phi_k \cdot \text{ plays the role of the electrostatic potential and } A_k \cdot \text{ the role of a scalar form of the vector field.}
\]
potential. The sum $\alpha_0 + m_k \cdot$ is the dressed mass of the quasiparticles. This model shows the analogy between the behavior of the modes of the phononic structure of figure 1 with spatio-temporal modulation of elastic parameters and the Dirac equation in the presence of an electromagnetic field. The modulation is shown to be able to break particle-hole symmetry and tune the spinor part of the elastic wave function and therefore its topology. This analogy between classical mechanics and quantum phenomena offers new modalities for developing more complex functions and in particular functions that relate to the control of the phase of elastic waves.

We report in Figure 3, the calculated band structure of the modulated system using the method of spectral energy density (SED) [27].

![Figure 3: Band structure of the mechanical model system of figure 1 with sinusoidal modulation of the parameter $\alpha$ calculated using the Spectral Energy Density (SED) method.](image)

The band structure has lost its mirror symmetry about the origin of the Brillouin zone. For the positive frequencies represented here, two band gaps appear on the positive side of the wave number axis. Non-reciprocal propagation of elastic waves results from this asymmetry at the frequency corresponding to the gaps. For negative frequencies, the gaps appear for negative
wave numbers. Therefore, for a particular wave number the particle (positive frequency)/hole (negative frequency) symmetry has been broken.

4 Conclusions

We have illustrated with two examples of phononic structures topologically constrained symmetry breaking. These model phononic structures exhibit either or both intrinsic and extrinsic symmetry breaking mechanisms. The elastic modes of the intrinsic system are described by a Dirac-like equation and time reversal, parity and chiral symmetry are broken individually. Joint time reversal and parity symmetry is retained. Application of an external spatio-temporal modulation on that system leads to extrinsic effects on the symmetry of the modes. In that case particle/hole symmetry is broken as well as the joint time reversal-parity symmetry. These model phononic structures provide insight into mechanisms for symmetry breaking that could be applied to achieve non-reciprocity and topologically constrained elastic modes in more complex phononic crystals or acoustic metamaterials.

References